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Translated by C.A.M.

PMM U.S.S.R.,Vol.51,No.4,pp.467-475,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

THE STRONG INJECTION OF GAS INTO A SUPERSONIC FLOW WITH TURBULENT MIXING*

I.I. VIGDOROVICH

A strong distributed injection of gas into a supersonic stream through a permeable plate is examined when the boundary layer (BL) is pressed back from a streamlined surface and the blown gas in the inviscid boundary region is separated from the oncoming stream by a turbulent mixing layer (ML). A disconnection criterion for the turbulent BL on injection and a similarity rule reflecting the fact that the flow over the plate is dependent on conditions at the end of the permeable section are formulated. Universal curves for the pressure distribution and injection-layer depth are given and flow force characteristics are calculated. The applicability of the solution derived from a simpler flow model, in which the ML is replaced by a contact breaking surface, is established, with a corresponding correction for turublent mixing.

1. Formulation of the problem. We examine supersonic flow over a smooth plate positioned at zero angle of attack to the oncoming stream with the injection of gas through a permeable section of its surface. Gas is blown in evenly, perpendicular to the plate with constant flow rate q_w and gas temperature at the wall T_w. Let us assume that, as a result of injection, the BL is pressed back from the entire permeable surface so that the gas blown into the inviscid boundary region *1* (Fig.1) is separated from the outer flow by the turbulent ML which develops from the start of the permeable section. This flow diagram corresponds to experimental data, for example /1, 2/.

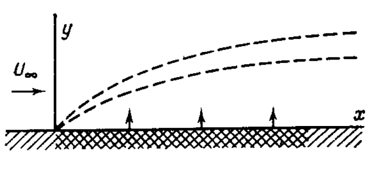

Fig.1

We denote by ε and δ the oreder of relative ML thickness and the inviscid part of the blowing layer. Since the BL is pressed back as a result of the blowing, the transverse component of the flow of mass in the boundary region in order of magnitude is not less than in the ML. The longitudinal component, however, is not greater than in the ML. Hence, and from the continuity equation it follows that $\delta \geqslant O(\varepsilon)$.

Let us examine the non-viscous part of the inflation layer. We shall use dimensionless variables. We will assign Cartesian coordinates to the length l of the permeable section, pressure to P_∞, density to $m_w P_\infty/(kT_w)$, velocity components to $\sqrt{kT_w/m_w}$ and the flow function to $lP_\infty\sqrt{m_w/(kT_w)}$, where k is Boltzmann's constant, and m_w is the molecular weight of the gas blown in; the parameters of the unperturbed oncoming flow are denoted by the subscript ∞. In accordance with the concept of a "thin layer" /2, 3/ we will assume that $\delta \ll 1$. At measured

*Prikl.Matem.Mekhan.,51,4,600-610,1987

supersonic speeds of the oncoming stream, according to linear theory, $p-1=O(\delta)$ in the blown layer. Hence, on the basis of adiabatic and Bernoulli integrals and the equation of continuity in the boundary region

$$\rho-1=O(\delta),\ u=O(\delta^{1/2}),\ v=O(\delta^{3/2}),\ \psi=O(\delta^{3/2})$$

From the transverse velocity and boundary condition estimates on the solid it follows that we can take $\delta=(q_w/P_\infty)^{2/3}(kT_w/m_u)^{1/3}$.

From the condition of equality of the orders of magnitude for the transverse velocity in the wall region and ML we obtain $\varepsilon=O(\delta^{3/2})$. In other words, when the BL is disconnected from the wall it is much shallower than the non-viscous part of the injection layer but the flow of gas in both layers is generally of the same order. Viscosity forces play a major role in the ML. Since the pressure perturbation is small and the flow in the boundary region is slow, in the limit as $\delta\to 0$ for the ML, the problem of isobaric mixing of the supersonic stream with the gas at rest, which has the physical parameters of the injected gas at the wall /3, 4/. If the BL thickness at the start of the injection is negligible, the problem is selfsimilar. The selfsimilarity of the velocity profile in the injection layer was observed in experiments /5/.

We denote by q_0 the flow of gas entering the selfsimilar ML from the region at rest. The quantity q_0/q_∞, where q_∞ is the supersonic flow rate, will depend generally on the relation between the temperatures and molecular weights of the injecting gas and the oncoming flow, the adiabatic indices, the Mach number and the Prandtl and Schmidt turbulent numbers. Given data on the empirical constants occurring in the set, a specific form of this dependence can be obtained from the solution of the corresponding set of equations for the ML. These data are very limited at present, however.

Assuming that the velocity pressure of the gas jets are entirely determined by the Mach number and the velocity pressure of the oncoming flow /6/, it is possible to use the following approximate formula:

$$q_0/q_\infty=b(M_\infty)\sqrt{m_wT_\infty/(m_\infty T_w)} \tag{1.1}$$

Such a dependence for $M_\infty=0$ was proposed /7/ as a result of calculations on the basis of the mixing path hypothesis. Apart from this, in the related problem of an axisymmetrical submerged jet, the proportionality of unified flow to the square root of the ratio of densities is well borne out by direct measurements /8/. The coefficient b diminishes as the Mach number increases. According to data /7/, $b\simeq 0.03$ for $M_\infty=0$ and is obviously several times less than this value at supersonic Mach numbers /9/.

In accordance with the estimates of gas-dynamic functions presented above, we introduce a new Mises independent variables x and $\Psi=\delta^{-3/2}\psi$ in the boundary region and we expand the dependent variables in asymptotic series in the small parameter δ. These asymptotic expansions and also the set of equations of the first approximation with boundary conditions on the plate and the outer boundary of the boundary region, adjacent to the ML, take the form /10/.

$$y\sim\delta Y+\ldots,\quad p\sim 1+\delta P+\ldots,\quad \rho\sim 1+\delta R+\ldots \tag{1.2}$$
$$u\sim\delta^{1/2}U+\ldots,\quad v\sim\delta^{3/2}V+\ldots$$

$$({}^1/_2U+P)_x=0,\quad P_\Psi=0,\quad (P-\gamma_wR)_x=0 \tag{1.3}$$
$$Y_\Psi=1/U,\quad Y_x=V/U$$
$$\Psi=-x,\quad 0\leqslant x\leqslant 1:\ Y=0,\quad U=0,\quad V=1,\quad R=P$$
$$\Psi=-q_0x/q_w,\quad 0\leqslant x\leqslant 1:\ P=\gamma_\infty M_\infty^2(M_\infty^2-1)^{-1/2}\times dY/dx$$

The equations of the thin layer (1.3) are obtained by substituting the expansions (1.2) into the Euler equation and the boundary condition at the outer boundary is specified according to the Ackeret formula for the pressure on the thin profile. In addition it is taken into account that the thickness of the displacement of the ML is negligible compared with the thickness of the whole injection layer.

2. The disconnection criterion of the BL. The similarity rule.

The expression for the flow function at the outer boundary of the non-viscous region may be rewritten in the form

$$\Psi=-\frac{B_*x}{B},\quad B=\frac{q_w}{q_\infty}\sqrt{\frac{m_\infty T_w}{m_wT_\infty}},\quad B_*=\frac{q_0}{q_\infty}\sqrt{\frac{m_\infty T_w}{m_wT_\infty}}$$

The non-viscous region close to the wall arises when the injection parameter is $B>B_*$. The quantity B_* can be considered as a critical value of the injection parameter at which the disconnection of the turbulent BL from the start of the permeable section takes place, where the thickness of the accumulating BL is negligible. If formula (1.1) is used for q_0, then the value $B_* = b(M_\infty)$ is only dependent on the Mach number of the oncoming stream.

Integration of thin-layer Eqs.(1.3) gives the following integral equation for the pressure on the plate /10/:

$$\int_{x/\omega}^{x} [P(\xi) - P(x)]^{-1/2} d\xi = \frac{\sqrt{2(M_\infty^2 - 1)}}{\gamma_\infty M_\infty^2} \times \int_0^x P(\xi)\, d\xi, \quad \omega = \frac{B}{B_*} \tag{2.1}$$

Since the velocity vector of the injected gas is directed perpendicular to the surface, in accordance with the linear theory of supersonic flow $P(0) = +\infty$. Changing in (2.1) to the independent variable $t = -(M_\infty^2 - 1)^{1/3}(\gamma_\infty M_\infty^2)^{-2/3}P$, we obtain the following set of equations for the functions $x(t)$ and $r(t)$

$$\int_{r(t)}^{t} \frac{x'(\tau)}{\sqrt{t-\tau}} d\tau = -\sqrt{2}\int_{-\infty}^{t} \tau x'(\tau)\, d\tau, \quad x[r(t)] = \frac{x(t)}{\omega} \tag{2.2}$$

which contains the single parameter ω which varies in the range $1 < \omega < \infty$. The relative injection parameter ω expresses the ratio of the gas flows travelling through the surface of the plate and the lower boundary of the ML. The condition $\omega \to 1$ corresponds to near-critical injection. The other limiting case $\omega = \infty$ occurs when the injection is fairly large, the flow of gas forced into the mixing zone is comparatively small and the ML can be taken to be the contact surface.

The function $x(t)$ in (2.2) is defined apart from an arbitrary constant. This indicates that it is essential to specify the additional boundary condition /3, 10/, for which the pressure at the termination point of injection can be set, for example. The solution of Eq. (2.2) $x_0(t, \omega)$ on the whole straight line $-\infty < t < +\infty$, which satisfies the boundary condition $x_0(+\infty, \omega) = 1$, has a universal character. Any solution of Eq.(2.2), which corresponds to a certain pressure at the end of the permeable section t_1, is expressed in terms of it by the formula $x(t) = x_0(t, \omega)/x_0(t_1, \omega), -\infty < t \leqslant t_1$. The universal solution itself $x_0(t, \omega)$ corresponds to the pressure distribution with a singularity where the injection is terminated $P(1) = -\infty$. This situation arises, for example, when the permeable plate ends in a step, for which a sufficiently low bottom pressure /3/ is established.

The above can be formulated as a similarity rule for the flow being studied and the corresponding distribution of selfinduced pressure on the plane can be written in the following parametric form:

$$\begin{aligned} x &= x_0(c_p', \omega)/x_0(c_{p1}', \omega), \quad c_p' = -2t \\ c_p &= (M_\infty^2 - 1)^{-1/4} B^{3/4} c_p' \end{aligned} \tag{2.3}$$

where c_{p1}' is the value of the pressure transformation coefficient at the injection termination point. For the distribution of the thickness of the injection layer y_δ and the force and moment coefficients we will have

$$\begin{aligned} y_\delta &= (M_\infty^2 - 1)^{1/4} B^{3/4} y_0(c_p', \omega)/x_0(c_{p1}', \omega) \\ c_y' &= (M_\infty^2 - 1)^{1/4} B^{-3/4} c_y = 2y_0(c_{p1}', \omega)/x_0(c_{p1}', \omega) \\ c_m' &= (M_\infty^2 - 1)^{1/4} B^{-3/4} c_m = 2z_0(c_{p1}', \omega)/x_0^2(c_{p1}', \omega) \\ y_0(t, \omega) &= -\int_{-\infty}^{t} \tau x_0'(\tau, \omega)\, d\tau, \quad z_0(t, \omega) = \int_{-\infty}^{t} x_0(\tau, \omega) y_0'(\tau, \omega)\, d\tau \end{aligned} \tag{2.4}$$

A simpler approximate form of the similarity rule (2.3) will be given in Sect.7.

3. The asymptotic form of the solution as $t \to -\infty$.

In the special case when $\omega = \infty$ we have $r(t) \equiv -\infty$ and the integral Eq.(2.2) has an analytical solution derived /11/ by the operational method. Its asymptotic expansion as $t \to -\infty$ takes the form* (*Vigdorovich I.I. Supersonic flow over solids with intense injection: Candidate Dissertation, Inst. Mech. Moscow State Univ., 1981.)

$$\begin{aligned} x_0(t, \infty) \sim \frac{\sqrt{2}}{2}(-t)^{-3/4} \exp\left(\frac{2t^3}{3\pi}\right) \times \\ \sum_{n=0}^{\infty}\left[\frac{\pi^n (6n+1)!!}{3(2n+1)!\, 6^{3n}} + \frac{2}{3}\left(\frac{3\pi}{4}\right)^n (2n-1)!! \sum_{m=0}^{n} \frac{(1/6)_m (5/6)_m}{(3/2)_m m!}\right] t^{3n} \end{aligned} \tag{3.1}$$

($(a)_m$ are Pochhammer's symbols /12/). We will generally seek the asymptotic presentation of the solution of Eq.(2.2) in an analogous form to (3.1)

$$x_0(t,\omega) \sim {}^1/_2 \sqrt{2} N(-t)^{-\beta} \exp(\alpha t^3)\,[1 - a_1 t^{-3} + a_2 t^{-6} + \ldots] \tag{3.2}$$

$$r(t) \sim t - b_1 t^{-2} + b_2 t^{-5} - b_3 t^{-8} + \ldots, \quad t \to -\infty \tag{3.3}$$

where the coefficients α, β, a_n, b_n are to be determined and N will be selected henceforth from the condition $x_0(+\infty, \omega) = 1$. Having found the logarithmic of the second relation (2.2), and substituting (3.2) and (3.3) into it and equating coefficients of identical powers and logarithms of t, we obtain

$$b_1 = \lambda(3\alpha)^{-1}, \quad b_2 = -\lambda(\lambda + \beta)(3\alpha)^{-2}$$

$$b_3 = ({}^5/_3\lambda^3 + {}^7/_2\beta\lambda^2 + \beta^2\lambda - 9\alpha\lambda a_1)(3\alpha)^{-3}, \quad \lambda = \ln\omega$$

From (3.2) we will have

$$x_0'(t,\omega) \sim {}^3/_2 \sqrt{2}\,\alpha N(-t)^{2-\beta} \exp(\alpha t^3) \times \left[1 - \left(a_1 + \frac{\beta}{3\alpha}\right)t^{-3} + \left(a_2 + \frac{\beta+3}{3\alpha}a_1\right)t^{-6} + \ldots\right] \tag{3.4}$$

$$y_0(t,\omega) \sim {}^1/_2 \sqrt{2} N(-t)^{1-\beta} \exp(\alpha t^3) \times \left[1 - \left(a_1 + \frac{1}{3\alpha}\right)t^{-3} + \left(a_2 + \frac{a_1}{3\alpha} - \frac{\beta+2}{9\alpha^2}\right)t^{-6} + \ldots\right] \tag{3.5}$$

Eq.(2.2) is conveniently transformed to the form

$$\sqrt{2}\,y_0(t,\omega) = \sqrt{t - r(t)} \int_0^1 x_0'[(t - r(t))\tau - t, \omega]\,\tau^{-1/2}\,d\tau \tag{3.6}$$

We expand both sides of Eq.(3.6), using (3.3)-(3.5), in a series up to terms of order t^{-6}. After integration from a comparison of the asymptotic expansions on both sides of the equation, for the required coefficients, we obtain

$$\alpha = \frac{2}{3\pi}(\operatorname{erf}\sqrt{\lambda})^{-2}, \quad \beta = \frac{3}{2} + \sqrt{\frac{\lambda}{\pi}}(\omega \operatorname{erf}\sqrt{\lambda})^{-1} \tag{3.7}$$

$$a_1 = -\frac{7\lambda}{24\omega^2} - \frac{17\pi}{24}\operatorname{erf}^2\sqrt{\lambda} - \frac{\sqrt{\pi}}{24\omega}\left(\frac{11}{3}\lambda + \frac{37}{2}\right)\operatorname{erf}\sqrt{\lambda}$$

A limit transition, as $\omega \to \infty$ for the coefficients (3.7) gives values corresponding to the expansion (3.1). According to (3.2), the pressure distribution has a weak logarithmic singularity at the start of the permeable section where $x = 0$.

4. The asymptotic form of the solution as $t \to +\infty$. The analytical solution of Eq.(3.2) has, as $t \to +\infty$, the asymptotic expansion /11/

$$x_0(t,\infty) \sim 1 - \sum_{n=0}^{\infty} \frac{\sqrt{2}\,(-\pi)^n (6n+1)!!}{(2n+1)!\,6^{2n+1}}\, t^{-3/2-3n} \tag{4.1}$$

In general, by analogy with (4.1), let us assume

$$x_0(t,\omega) = 1 - At^{-3/2} + O(t^{-9/2}), \quad t \to +\infty \tag{4.2}$$

We introduce the notation $R = r(+\infty)$, $Y = y_0(R,\omega)$, $w = t - R$. Substituting (4.2) into the second relation of (2.2), we will have

$$\varepsilon(w) \equiv R - r(w) = \frac{Aw^{-3/2}}{\omega x_0'(R,\omega)} + O(w^{-5/2}), \quad w \to +\infty \tag{4.3}$$

Let us rewrite the integral Eq.(2.2) in the form

$$\int_0^w \frac{x_0'(R+\tau,\omega)}{\sqrt{2(\omega-\tau)}}\,d\tau + g(w) = Y - \int_0^w (R+\tau)\,x_0'(R+\tau,\omega)\,d\tau \tag{4.4}$$

$$g(w) = \int_0^1 \frac{x_0'(R - \varepsilon(w)\tau,\omega)}{\sqrt{2(w + \varepsilon(w)\tau)}}\,d\tau, \quad w \geqslant 0 \tag{4.5}$$

$$g(w) = \frac{A\sqrt{2}}{2\omega w^2} + O(w^{-3}), \quad w \to +\infty \tag{4.6}$$

The asymptotic representation (4.6) is obtained as a result of substituting (4.3) into integral (4.5). We apply a Laplace transformation /13/ to the integral Eq.(4.4).

Let us denote by $X(z), G(z)$ of the Laplace transforms of the functions $x_0'(R+w, \omega)$ and $g(w)$ respectively. By the convolution theorem /13/ for $X(z)$ we obtain an ordinary differential equation of the first order, whose solution is

$$X(z) = \Big[C + \int_0^z (\zeta G(\zeta) - Y) H(\zeta)\, d\zeta\Big] \Big/ H(z)$$

$$C = \int_0^\infty (Y - zG(z)) H(z)\, dz, \quad H(z) = \exp\left(-\frac{\sqrt{2\pi}}{3} z^{3/2} - Rz\right)$$

Now the function $x_0(t, \omega)$, using the integral of the Mellin transform /13/ can be represented in the form

$$x_0(t, \omega) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} Z(z) \exp(zt) \frac{dz}{z}, \quad \sigma > 0, \quad t \geqslant R \tag{4.7}$$

$$Z(z) = (X(z) + x_0(R, \omega)) \exp(-Rz)$$

The asymptotic representation (4.6) enables us to write /14/

$$G(z) = G_0 + {}^1/_2 \sqrt{2} A \omega^{-1} z \ln z + G_1 z + O(z^2 \ln z), \quad z \to 0$$

where G_0 and G_1 are certain constants which depend on the behaviour of the function $g(w)$ over the whole semi-axis $0 \leqslant w < +\infty$. We expand the function $Z(z)$ in an asymptotic series as $z \to 0$

$$\begin{aligned} Z(z) = {} & C + 1/\omega - (Y + R/\omega) z + {}^1/_3 C \sqrt{2\pi} z^{3/2} + \\ & {}^1/_2 (G_0 + RY + R^2/\omega) z^2 - {}^1/_5 Y \sqrt{2\pi} z^{5/2} + \\ & {}^1/_6 A \sqrt{2} z^3 \ln z/\omega + {}^1/_3 (G_1 - G_0 R - {}^1/_2 Y R^2 - \\ & {}^1/_6 A \sqrt{2}/\omega + {}^1/_9 C\pi - {}^1/_6 R^3/\omega) z^3 + {}^1/_3 \sqrt{2\pi} (1/_2 RY + \\ & {}^3/_{14} G_0) z^{7/2} + O(z^4 \ln z) \end{aligned} \tag{4.8}$$

Here, it is taken into account that in view of the definition of R and the second relation (2.2) $x_0(R, \omega) = 1/\omega$. Now let us substitute (4.8) into integral (4.7) and change the order of integration and summation. As a result, we obtain the asymptotic expansion of the universal function $x_0(t, \omega)$ as $t \to +\infty$ /13/. The terms of the sum (4.8) with integer powers of z do not contribute to the asymptotic form of the original. On comparing the asymptotic series derived in this way with (4.2), we determine the required coefficients. Finally we will obtain

$$\begin{aligned} x_0(t, \omega) = {} & 1 - \frac{(\omega - 1)\sqrt{2}}{6\omega} t^{-3/2} - \frac{3\sqrt{2}}{20} Y t^{-5/2} + \\ & \frac{\omega - 1}{9\omega^2} t^{-3} + O(t^{-7/2}), \quad y_0(t, \omega) = \frac{(\omega - 1)\sqrt{2}}{2\omega} t^{-1/2} + \\ & \frac{\sqrt{2}}{4} Y t^{-3/2} - \frac{\omega - 1}{6\omega^2} t^{-2} + O(t^{-5/2}), \quad t \to +\infty \end{aligned} \tag{4.9}$$

The quantity Y remains undetermined. Since $Y \to 0$ as $\omega \to \infty$, in the limit the first relation (4.9) is identical with (4.1). The second asymptotic expansion (4.9) is obtained from the first on the basis of (2.4).

5. Near-critical injection. To study the asymptotic form of the solution as $\omega \to 1$, it is convenient to use Eq.(2.1), rewritten in the form

$$\sqrt{2}\, y_0(x_0, \omega) = \int_{x_0/\omega}^{x_0} [y_0'(\xi, \omega) - y_0'(x_0, \omega)]^{-1/2} d\xi \tag{5.1}$$

We will seek the solution of (5.1) in the form of the following asymptotic series:

$$y_0(x_0, \omega) = \mu^{1/3} [\eta_1(x_0) + \mu \eta_2(x_0) + O(\mu^2)], \quad \mu = 1 - 1/\omega \tag{5.2}$$

Let us change in (5.1) to the integration variable τ according to the formula $\xi - x_0 = -\mu x_0 \tau^2$ and expand the difference under the root in a Taylor series at the point x_0

$$\begin{aligned} y_0'(\xi, \omega) - y_0'(x_0, \omega) = {} & \mu^{1/3} [-\mu x_0 \tau^2 \eta_1''(x_0) + \\ & {}^1/_2 \mu^2 x_0^2 \tau^4 \eta_1'''(x_0) - \mu^2 x_0 \tau^2 \eta_2''(x_0) + O(\mu^3)] \end{aligned}$$

Substituting this expansion into (5.1) and equating coefficients of the same powers of the small parameter, we obtain the ordinary differential equations for the required functions

$$\eta_1 \sqrt{-\eta_1''} = \sqrt{2x_0}, \quad 2\eta_2 (-\eta_1'')^{3/2} = \sqrt{2x_0} (\eta_2'' - {}^1/_6 x_0 \eta_1''') \tag{5.3}$$

By reducing the order of the equation of the first approximation by means of the substitution /15/

$$x_0 = \exp\left(\int^u \frac{du}{z(u)}\right), \quad \eta_1 = u x_0(u) \tag{5.4}$$

we obtain the Abel equation of the second kind

$$zz' + z + 2u^{-2} = 0 \tag{5.5}$$

If the solution of Eq.(5.5) is known, the following terms of series (5.2) may be obtained in quadratures. Indeed, the equation of the second approximation (5.3), after substituting $\eta_2 = v(u)\, x_0(u)$, becomes

$$u^3 z^2(u)\, v'' - 2uv' - 4v = {}^1/_3 u + {}^2/_3 z(u) \tag{5.6}$$

and the corresponding equations for the terms of expansion (5.2) of higher order will only differ from (5.6) on the right-hand side. Now it is sufficient to note that the function $z(u)$ is a partial solution of the uniform Eq.(5.6), after which its common integral can be written in the form /15/

$$v = z(u)\left\{\int^u \left[\int^{u_1} \frac{2}{3} z(u_2)\, x_0(u_2) \frac{du_2}{u_2^3}\right] \frac{du_1}{z^3(u_1)\, x_0(u_1)}\right\} - \frac{u}{18}$$

The following series of transformations:

$$z = -u - 2w, \quad u' + u^3 + 2wu^2 = 0 \tag{5.7}$$

$$\xi = w^2 - 1/u, \quad w' = \xi - w^2 \tag{5.8}$$

converts (5.5) into the special Ricatti Eq.(5.8) /15/ which, in its turn, after replacement of the variable $w = (\ln A(\xi))'$, reduces to the Airey equation $A'' = \xi A$ /16/. From (5.7), (5.8), (5.4) we will have

$$u = -1/w', \quad z = 1/w' - 2w = (\ln w')'$$

$$\int^u \frac{du}{z(u)} = \int^{\xi} \frac{d\xi}{w'(\xi)} = 2\ln A(\xi) + \ln w'(\xi)$$

$$x_0 = -w'(\xi)\, A^2(\xi) = A'^2(\xi) - \xi A^2(\xi), \quad \eta_1 = A^2(\xi) \tag{5.9}$$

The general solution of the Airey equation is a linear combination of Airey functions of the first and second kind $Ai(\xi), Bi(\xi)$ /16/. We will determine the range of variation of the parameter ξ and constants of integration arising from the physical conditions of the problem. As follows from (5.9), at the point ξ_0, for which $x_0 = 0$, $\eta_1 = 0$, we must have $A(\xi_0) = 0$, $A'(\xi_0) = 0$. These conditions cannot be satisfied for a finite value of ξ_0 since the Airey equation has no singularities in a finite plane. The Airey functions oscillate as $\xi \to -\infty$, so that $x_0(\xi)$ does not tend to zero. As $\xi \to +\infty$, the function $Ai(\xi)$ decays exponentially, while $Bi(\xi)$ increases, whence it follows that $A(\xi) = DAi(\xi)$, and D is a constant. We define D and ξ_1 the left limit of the range of variation of ξ, from the conditions $x_0(\xi_1) = 1$, $\eta_1(\xi_1) = 0$ This gives $\xi_1 \simeq -2.33811$ as the largest root of the Airey function of the first kind, $D = 1/A'i(\xi_1) \simeq 1.4262$ /17/.

Now the limit pressure distribution, as $\omega \to 1$ and the injection layer thickness distribution can be written in the following parametric form:

$$c_p' = -4\mu^{1/3} \frac{A'i(\xi)}{Ai(\xi)} + O(\mu^{4/3}) \tag{5.10}$$

$$y_0 = \mu^{1/3}[DAi(\xi)]^2 + O(\mu^{4/3})$$

$$x_0 = D^2[Ai'^2(\xi) - \xi Ai^2(\xi)], \quad \xi_1 \leqslant \xi < +\infty$$

Hence, using the asymptotic form of the Airey function as $\xi \to +\infty$ /16/, we obtain the asymptotic representation of the universal function, which is correct when $\omega = 1, s \to +\infty$

$$x_0 \sim \frac{D^2}{2\pi} s^{-2} \exp\left(\frac{1}{6}s^3 + 1\right)\left(1 + \frac{14}{3}s^{-3} + \frac{470}{9}s^{-6} + \ldots\right), \qquad s = \mu^{-1/3} t \tag{5.11}$$

This result agrees completely with (3.2) and (3.7), since the limit transition $\omega \to 1$ in (3.7) gives $\mu\alpha = {}^1/_6, \beta = 2, a_1\mu^{-1} = -14/3$. In addition, comparing expansions (3.2) and (5.11) we will have

$$N = \nu\mu^{\beta/3}, \quad \nu = \frac{D^2 e\sqrt{2}}{2\pi} \quad (1 + O(\mu)), \quad \omega \to 1 \tag{5.12}$$

We note that the asymptotic representation of the universal function as $\omega \to 1$, $s \to +\infty$, which can be obtained from (5.10), and the expressions of the Airey function at zero, are not identical with the limit form of expansion (4.9) as $\omega \to 1$. One reason for this is the non-uniform character of the asymptotic series (5.2) in the neighbourhood of the point $x_0 = 1$.

6. Numerical solution of the integral equation. Using the independent variable s, we can rewrite Eq.(2.2) integrally in the form

$$\int_{-\infty}^{s} \frac{x'(\tau)}{\sqrt{s-\tau}}\,d\tau = \int_{-\infty}^{s} \left\{\frac{1}{\omega}[s-q(\tau)]^{-1/2} - \sqrt{2\mu}\,\tau\right\} x'(\tau)\,d\tau$$

$$s = \mu^{-1/3}t, \quad q(s) = \mu^{-1/3}r(t)$$

and we can solve it for the function $x(s)$ on the left-hand side, as an Abel equation /13/. As a result, we obtain

$$x(s) + \int_{-\infty}^{s} x(\tau)\,dK(s,\tau,q(\tau)) = 0, \quad x[q(s)] = \frac{x(s)}{\omega} \tag{6.1}$$

$$K(s,\tau,q(\tau)) = \frac{2}{\pi\omega}\arcsin\sqrt{\frac{s-\tau}{s-q(\tau)}} - \frac{2}{\pi}\tau\sqrt{2\mu(s-\tau)}$$

For a fairly large negative number, in absolute magnitude, a, we can write approximately

$$x(s) + \int_{a}^{s} x(\tau)\,dK(s,\tau,q(\tau)) = -\int_{-\infty}^{a} E(\tau)\,dK(s,\tau,q(\tau)) \tag{6.2}$$

$$E(s) = {}^1/_2\sqrt{2}\,(-s)^{-\beta}\exp(\alpha\mu s^3)(1 - a_1\mu^{-1}s^{-3})$$

Here, the function $E(s)$ is chosen in accordance with the asymptotic expansion (3.2) and formula (5.12); the function $q(s)$ on the right-hand side of (6.2) is specified by the asymptotic series (3.3). The error, as a result of the replacement of the solution $x(s)$ by its approximate value $E(s)$ can be evaluated by the first discarded term of the asymptotic form (3.2), which is equal to $a_2/(\mu^2 s^6)$ also taking into account that, in accordance with (3.1), (5.11) we obtain $a_2\mu^{-2} = 1801\pi^2/1152$ for $\omega = \infty$ and $a_2\mu^{-2} = 470/9$ for $\omega = 1$. In order that the unknown function should be of the order of unity we will use the change of variable $x(s) = E(s)\chi(s)$, in the interval $[a, b]$ where the function $E(s)$ is sufficiently close to the solution.

We shall solve Eq.(6.2) as a Volterra integral equation of the second kind, replacing the integral by a finite sum. For example, for $a < s \leqslant b$ we will use the following quadratic formula of the first algebraic degree of accuracy with the twofold expansion of the residue from the Euler quadrature /18/:

$$\int_{a}^{s_n} \chi(\tau)P(s_n,\tau)\,d\tau \approx \sum_{k=1}^{m} A_{2k-1}\chi(s_{2k-1}) + C_0[\chi(s_n)-\chi(a)] + C_1(\chi'(s_n)-\chi'(a)] \tag{6.3}$$

$$P(s,\tau) = E(\tau)\frac{d}{d\tau}K(s,\tau,q(\tau))$$

$$A_k = \int_{s_{k-1}}^{s_{k+1}} P(s_n,\tau)\,d\tau, \quad C_i = \frac{(s_n-a)^i}{(i+1)!}\left[\int_{a}^{s_n} B_{i+1}\left(\frac{\tau-a}{s_n-a}\right)P(s_n,\tau)\,d\tau - \right. \tag{6.4}$$

$$\left.\sum_{k=1}^{m} A_{2k-1}B_{i+1}\left(\frac{s_{2k-1}-a}{s_n-a}\right)\right], \quad i = 0,1; \quad s_n = a + nh$$

Here $B_i(x)$ are Bernoulli polynomials /18/ and h is the integration step. The derivative of the unknown function in (6.3) is approximated by a finite difference. Formulae (6.3) and (6.4) are written for $n = 2m$ and for odd values of n are analogous.

Now it is possible to define sequentially, in recurrent formulae, the values $\chi(s_n)$, $x(s_n)$ in each interval, and then $q(s_n)$ from the second equation of (6.1).

ω	1	1,1	1,5	2	4	∞
ν	1,2444	1,22	1,21	1,17	1,11	1
$-c_{m1}^{\parallel}\cdot 10^3$	834,9	850	893	899	902	957

When evaluating the integrals (6.4), the function $q(s)$ was interpolated by a third-degree polynomial.

To normalize the numerical solution, i.e. to define the value of ν in (5.12) for each ω for sufficiently large s we take advantage of the asymptotic expansion (4.9), into which we substitute $x_0(t,\omega)=\nu x(s)$, $Y=\nu y(q(s))+O(s^{-3/2})$. The values of ν for various injection parameters are given in the table.

The accuracy of the method described was checked on limit analytical solutions. Thus, for $\omega=\infty$, $a=-5$, $b=-1.8$, $h=0.01$, the numerical solution differs from the exact solution /11/ by less than 0.1%.

7. Results of calculations and conclusions. It is convenient to represent the similarity variables in the form

$$c_p'=\mu^{1/3}c_p'',\quad y_0=\mu_0^{1/3}\eta_0,\quad c_y'=\mu^{1/3}c_y'',\quad c_m'=\mu^{1/3}c_m''$$
$$(\mu=1-1/\omega)$$

Universal curves are presented in Fig.2, which give the distribution along the length of the plate of the transformed pressure coefficient c_p'' (the solid lines) and the thickness of the injection layer η_0 (the dotted lines) for different values of the parameter ω. To obtain the pressure distribution and the distribution of injection layer thickness which correspond to a certain pressure coefficient at the termination point of injection c_{p1}'', the solid curves in Fig.2 should be stretched along the abscissa axis and the dotted curves along the ordinate axis $1/x_0(c_{p1}'',\omega)$, once, and their parts corresponding to the interval $0\leqslant x\leqslant 1$ should be taken. The dependences of the force coefficients (the solid lines) and the moment (the dotted lines) on the pressure at the end of the permeable section are given in Fig.3. In Figs.2, 3 curves *1-3* correspond to the values of the relative injection parameter $\omega=1;1.5;\infty$ respectively. The excess force acting on the plate, as expected from (2.4), (4.9), tends to zero as $c_{p1}''\to-\infty$, in other words, at low pressure at the end of the permeable section the regions with increased and reduced pressure almost cancel each other out. The moment coefficient c_m'' calculated with respect to the origin of coordinates, tends towards the negative value c_{m1}'' as $c_{p1}''\to-\infty$, whose modulus is equal to twice the area under the corresponding curve $\eta_0(x_0)$ in Fig.2. The value c_{m1}'' for various injection parameters is given in the table.

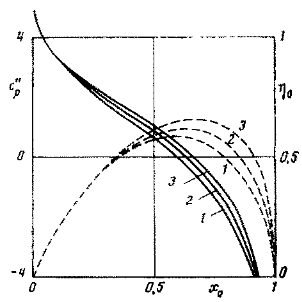

Fig.2

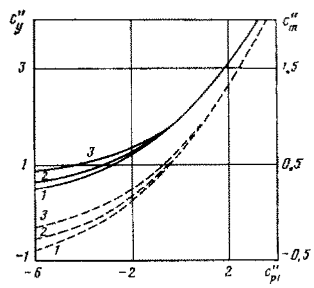

Fig.3

In Fig.2, the similarity curves which relate to the limit values of the injection parameter 1 and ∞ are close to one another. Specific dependences of the pressure on the longitudinal coordinate for fixed c_{p1}'' and various ω are obtained by stretching corresponding sections of the universal curves along the abscissa axis, where their ends coincide, while the graphs of these dependences move even closer together. Consequently, at identical pressure values at the termination point of the injection c_{p1}'' the distribution of the transformed pressure coefficient c_p'' along the length of the plate is practically independent of the injection parameter. This makes it possible to give a simpler formulation of the similarity rule (2.3):

$$x=x_0(c_p'')/x_0(c_{p1}''),\quad c_p=(M_\infty^2-1)^{-1/3}[B(B-B_*)]^{1/3}c_p'' \tag{7.1}$$

Thus, the results for the pressure distribution obtained in an ideal formulation, when the ML is examined as a contact-breaking surface, are suitable over the whole of the supercritical range of injection parameter if c_p' is replaced by c_p''. For large injections c_p'' is little different from c_p'. In this case, as demonstrated in /2/, the similarity rule (7.1) and the calculated dependences presented in Fig.2 are in good agreement with experimental data.

Permitted limits of the change in the injection parameter are defined, on the one hand, by the condition for the ML thickness to be small compared with the thickness of the inviscid boundary region and on the other, by the validity of the thin-layer approximation and of the linear theory of supersonic flow. For typical thickness of the ML and the non-viscous part of the injection layer it is possible to take lq_0/q_∞ and $l(M_\infty - 1)^{1/4}[B(B - B_*)]^{1/3}$ respectively. Then the limits on the defining parameters of the problem are formulated as

$$[\omega(\omega - 1)/B_*]^{1/3}(M_\infty^2 - 1)^{1/4} \gg \sqrt{m_w T_\infty/(m_\infty T_w)}$$

$$[B(B - B_*)]^{1/3}(M_\infty^2 - 1)^{1/3} \ll 1, \quad [B(B - B_*)]^{1/3}(M_\infty^2 - 1)^{-4/3} \ll 1$$

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Translated by C.A.M.